You should show your work. You will lose points if you do not show your work. The question might contain less or more than enough information. If the question does not contain all the necessary informations, you are supposed to make realistic assumptions. You will lose points if you make unnecessary assumptions.

**Discussion**

Answer the following question with words only. You do not need to give a quantitative answer, a qualitative answer is enough. You will lose points if you use equations.

1. Consider a uniformly charged rectangular object. When this object is placed in water, it is observed that the surface on top of the object is elevated. Explain this phenomena both from (i) a macroscopic point of view in terms of the forces acting on dielectrics with dielectric constant $\epsilon$ (10 points) and from (ii) a microscopic point of view in terms of the forces acting of individual dipoles making the water (10 points)

**Short Questions**

You can give short answers in the following questions. You do not need to show your work in detail.

2. Consider a conducting sphere of radius $R$. Assume that the sphere is covered by a dielectric material with dielectric constant $\epsilon$. and thickness $R$. Starting from $\nabla \cdot \vec{D} = 4\pi\rho$ and $\vec{D} = \epsilon\vec{E}$, calculate the electric field everywhere, i.e. at $r < R$, $R < r < 2R$ and $r > 2R$. (10 points)

3. Consider two point charges $q_1$ and $q_2$ in front of an infinite conducting plane. Denote the positions of the charges by $\vec{r}_1$ and $\vec{r}_2$ respectively. What is the force acting on the charge $q_1$? (10 points)

4. Consider an infinite plane lying in the $xy$ plane. The electrostatic potential of an arbitrary point on the plane is known to be $V(x, y)$. Write down an integral expression that will give the electrostatic potential at an arbitrary point $\vec{r} = (x, y, z)$. (10 points)

**Explicit Calculations:**

In answering following problems, show your steps in detail. You should also explain why you do a specific step.

5. Consider an object with a uniform charge density given by $\rho_0$. Assume that the shape of this object is given by $r(\theta, \phi) = r_0(1 + aY_{10}(\theta, \phi))$ (Assume a is sufficiently small so that $r(\theta, \phi)$ is always positive)

(a) Sketch what the object looks like. (10 points)

(b) Calculate all the monopole and dipole moments of the object. (10 points)
6. Using the uniqueness of the solution to the Poisson’s equation with given boundary conditions, prove that

\[
\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_l'}{r'_l+1} Y_{lm}(\theta', \phi') Y_{lm}(\theta, \phi)
\]

where \(\theta, \phi\) and \(\theta', \phi'\) are the angular coordinates of the vector \(\vec{r}(\vec{r}')\), \(r_\leq = \min(r, r')\) and \(r_\geq = \max(r, r')\) For this purpose, follow the following steps:

(a) Prove that \(V(\vec{r}, \vec{r}') = 1/|\vec{r} - \vec{r}'|\) satisfies the Poisson equation

\[
\nabla^2 V = -4\pi \delta^3(\vec{r} - \vec{r}')
\]

where the derivatives are with respect to the components of the vector \(\vec{r}\), and vanishes when \(\vec{r} \to \infty\), which is nothing but the equation governing the potential of a unit point charge at the point \(\vec{r}'\) (10 points)

(b) Divide the space in two regions: inside the sphere of radius \(r'\) (region I) and outside the sphere (region II). Inside both of these regions \(\delta(\vec{r} - \vec{r}') = 0\), hence inside these regions \(V(\vec{r}, \vec{r}')\) satisfies the Laplace equation. Write the Laplace equation in polar coordinates, and assume a solution of the form

\[
V(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi)
\]

What are the equations satisfied by the functions \(R(r)\), \(\Theta(\theta)\) and \(\Phi(\phi)\)? (10 points)

The general solutions of these equations can be written as

\[
V_{lm}(r, \theta, \phi) = \left( a_{lm} r^l + \frac{b_{lm}}{r^{l+1}} \right) Y_{lm}(\theta, \phi)
\]

for \(l = 0, 1, \cdots\) and \(m = -l, \cdots, l\), where \(a_{lm}\) and \(b_{lm}\) are arbitrary coefficients. Thus the most general solution of the Laplace’s equation in spherical coordinates can be written as the sum of such terms.

(c) Write down the most general solution in region I and in region II consistent with the boundary conditions that the solution has to be finite as \(r \to 0\) and \(r \to \infty\). (10 points)

(d) What are the boundary conditions at the boundary \(r = r'\)? (10 points)

(e) Any function of \(\theta\) and \(\phi\) can be written in terms of spherical harmonics. Considering that the surface charge density at the surface \(r = r'\) can be written as

\[
\sigma(\theta, \phi) = \delta(\cos \theta - \cos \theta') \delta(\phi - \phi')
\]

expand it in terms of spherical harmonics. (10 points)

(f) Using the expansion obtained in the previous part in the expression for the boundary conditions, obtain the values for the unknown coefficients in the general expansion of the solution. (10 points)